

GRAVITATIONAL WAVES SPECTRUM IN SQUEEZED VACUUM STATE

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Abstract

Gravitational waves are placed in the squeezed vacuum state and obtained its spectrum for the expanding flat FLRW universe. The gravitational wave spectrum gets enhanced due to the squeezing effect and is likely to be detected with the Einstein Telescope. The spectral energy density of gravitational waves in the squeezed vacuum state does not exceed the nucleosynthesis upper bound.

1 Introduction

The general theory of relativity predicts the existence of gravitational waves (GWs). There is a tremendous progress in understanding the various aspects of the GWs over the several decades due to theoretical as well as observational techniques. There are many potential candidates which can generate gravitational waves that range from astrophysical systems such as the binary systems of black hole mergers, neutron stars, and core collapse of supernovae [1] to cosmic inflation [2, 3]. Inflation also predicts a nearly scale invariant spectrum for the scalar and tensor perturbations which occurred in the early universe. The tensor perturbations of cosmological in origin are known as primordial gravitational waves.

The primordial gravitational waves are expected to carry information on the physical conditions of the early universe. The gravitational waves have been propagating through various evolution stages of the universe since the inflationary period. Therefore the spectrum of these waves, to be observed today, depends on the behavior of all the evolutionary stages, including the current accelerating stage.

The gravitational waves were generated during the inflation period due to the quantum fluctuations of vacuum [4]. Initially, there may be no gravitational waves (vacuum state), but they can be generated later on and evolved into a multi-particle quantum state called the squeezed vacuum state [5], a well-known state in the quantum optics [6, 7, 8]. Therefore the primordial gravitational waves are expected to exist in the squeezed vacuum state [9, 10, 11]. If this is the fact, then the squeezing effect must have an observation consequence such as the effect be reflected on the amplitude of the gravitational waves spectrum to be observed today. However this aspected is not examined so far in the context of any ongoing mission that is likely to detect the gravitational waves and hence the present study.

The aim of the present work is to obtain the amplitude of the gravitational wave spectrum by considering them in the squeezed vacuum state for the expanding flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe, and hence to seek the possibility of detecting the modified amplitude of gravitational waves spectrum with the sensitivity of various gravitational wave detectors [12, 13, 14, 15, 16]. Also to estimate the spectral energy density of gravitational waves in the accelerating flat FLRW universe.

2 Expansion history of the universe

In the standard cosmology, homogeneous, isotropic and expanding universe can be described with the FLRW metric. The scale factor, a , which is a parameter in the FLRW metric, depends on each evolution era of the universe. The various evolution stages of the universe can be summarized

in terms of the scale factor as follows:

The initial inflationary (*i*) stage:

$$a(\eta) = l_0 |\eta|^{1+\beta}, \quad -\infty < \eta \leq \eta_1, \quad (1)$$

where l_0 and β are arbitrary constants, $1 + \beta < 0$ and $\eta_1 < 0$. The case $\beta = -2$ denotes the exact de Sitter expansion of inflation. $d\eta = \frac{dt}{a}$ is the conformal time.

The reheating (*z*) stage [17]

$$a(\eta) = a_z (\eta - \eta_P)^{1+\beta_s}, \quad \eta_1 \leq \eta \leq \eta_s, \quad (2)$$

where $\beta_s + 1 > 0$. β_s describes the expansion behavior of the reheating stage.

The radiation-dominated (*e*) stage:

$$a(\eta) = a_e (\eta - \eta_e), \quad \eta_s \leq \eta \leq \eta_2. \quad (3)$$

The matter-dominated (*m*) stage:

$$a(\eta) = a_m (\eta - \eta_m)^2, \quad \eta_2 \leq \eta \leq \eta_E. \quad (4)$$

The accelerating stage up to the current stage:

$$a(\eta) = l_H |\eta - \eta_a|^{-1}, \quad \eta_E \leq \eta \leq \eta_H, \quad (5)$$

where l_H is the Hubble radius at present,

$$l_H = \left(\frac{a^2}{a'} \right)_{\eta_H} = \frac{1}{H}.$$

The continuous joining of the functions $a(\eta)$ and $a'(\eta)$ at the points of transition η_1 , η_s , η_2 and η_E provide the link between the arbitrary constants between eq. (1) to eq. (5).

3 Basic gravitational wave equation

The FLRW metric, which is a solution of the Einstein's field equations, serve as the basic equation for the primordial gravitational waves.

The perturbed metric for a flat FLRW universe can be written as

$$ds^2 = a^2(\eta) [-d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j], \quad (6)$$

where h_{ij} is a transverse-traceless tensor perturbation and δ_{ij} is the flat space metric, $|h_{ij}| \ll \delta_{ij}$, and $\partial_i h^{ij} = 0, \delta^{ij} h_{ij} = 0$.

In quantum theory, the field $h_{ij}(\mathbf{x}, \eta)$ can be written in Fourier mode as

$$h_{ij}(\mathbf{x}, \eta) = \frac{C}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} \frac{d^3\mathbf{k}}{\sqrt{2k}} \sum_{p=1}^2 \left[h_k^{(p)}(\eta) c_k^{(p)} e^{i\mathbf{k}\cdot\mathbf{x}} \varepsilon_{ij}^{(p)}(\mathbf{k}) + h_k^{(p)*}(\eta) c_k^{(p)\dagger} e^{-i\mathbf{k}\cdot\mathbf{x}} \varepsilon_{ij}^{(p)*}(\mathbf{k}) \right], \quad (7)$$

where $C = \sqrt{16\pi} l_{pl}$ is the normalization constant, $l_{pl} = \sqrt{G}$ is the Planck length and \mathbf{k} is wave vector. The wave number is $k = (\delta_{ij} k^i k^j)^{\frac{1}{2}}$ and is related to wavelength, $\lambda = \frac{2\pi a}{k}$.

The two polarization states $\varepsilon_{ij}^{(p)}$, $p = 1, 2$ are symmetric and transverse-traceless and satisfy the conditions $\varepsilon_{ij}^{(p)} \delta^{ij} = 0$, $\varepsilon_{ij}^{(p)} k^i = 0$, $\varepsilon_{ij}^{(p)} \varepsilon^{(p')ij} = 2\delta_{pp'}$, $\varepsilon_{ij}^{(p)}(-\mathbf{k}) = \varepsilon_{ij}^{(p)}(\mathbf{k})$. These linear polarizations are called plus (+) polarization and cross (\times) polarization.

The creation and annihilation operators $c_k^{(p)\dagger}$ and $c_k^{(p)}$ satisfy the relationships

$$[c_k^{(p)}, c_{k'}^{(p')\dagger}] = \delta_{pp'} \delta^3(k - k'),$$

$$[c_k^{(p)}, c_{k'}^{(p')}] = [c_k^{(p)\dagger}, c_{k'}^{(p')\dagger}] = 0.$$

The evolution of creation and annihilation operators are governed by the Heisenberg equations of motion

$$\frac{d}{d\eta} c_k^\dagger(\eta) = -i[c_k^\dagger(\eta), H], \quad (8)$$

$$\frac{d}{d\eta} c_k(\eta) = -i[c_k(\eta), H]. \quad (9)$$

The initial vacuum state $|0\rangle$ is defined as

$$c_k^p|0\rangle = 0.$$

The Bogoliubov transformations for eq. (8) and eq. (9) are

$$c_k^\dagger(\eta) = u_k^*(\eta) c_k^\dagger(0) + v_k^*(\eta) c_k(0), \quad (10)$$

$$c_k(\eta) = u_k(\eta) c_k(0) + v_k(\eta) c_k^\dagger(0), \quad (11)$$

where $c_k^\dagger(0)$ and $c_k(0)$ are the initial values of the operators, $u_k(\eta)$ and $v_k(\eta)$ are complex functions and they satisfy the condition

$$|u_k|^2 - |v_k|^2 = 1.$$

The coupling of the mode functions $h_k(\eta)$ with $a(\eta)$ gives

$$h_k^{(p)} = \frac{\mu_k^{(p)}}{a}. \quad (12)$$

The mode functions can have the following form

$$\mu_k(\eta) = u_k(\eta) + v_k^*(\eta), \quad (13)$$

which satisfies the equation of motion

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right) \mu_k = 0, \quad (14)$$

where prime (') indicates the derivative with respect to the conformal time η . Using eq. (12), eq. (14) becomes

$$h_k'' + 2\frac{a'}{a}h_k' + k^2 h_k = 0. \quad (15)$$

For each wave number k and polarization $p = 1, 2$, the mode functions $h_k^{(p)}$ satisfy eq. (14). Hence, from here onwards, we take the contribution from each polarization to be the same.

There are two limiting cases for eq. (14), *viz.* $k^2 \gg \frac{a''}{a}$ and $k^2 \ll \frac{a''}{a}$. For short wavelength limit, $k^2 \gg \frac{a''}{a}$, the wave is outside the potential barrier and does not interact with the barrier, and propagates with an adiabatically decreasing amplitude $h_k(\eta) \propto \frac{1}{a(\eta)}$. For long wavelength limit, $k^2 \ll \frac{a''}{a}$, the wave is inside the potential barrier. The wave interacts with the barrier and gets amplified above $h_k(\eta) \propto \frac{1}{a(\eta)}$. At the same time, a wave propagating in the opposite direction is created, giving rise to stochastic standing waves.

3.1 Amplitude of gravitational waves

The power spectrum of the gravitational waves is defined by the variance of the field:

$$\langle 0 | h_{ij}(\mathbf{x}, \eta) h^{ij}(\mathbf{x}, \eta) | 0 \rangle = \int_0^\infty h^2(k, \eta) \frac{dk}{k}, \quad (16)$$

where

$$h^2(k, \eta) = \frac{C^2}{2\pi^2} k^2 \sum_{p=1,2} |h_k^{(p)}(\eta)|^2, \quad (17)$$

gives the mean-square value of the gravitational waves with interval k . Also, [18]

$$h^2(k, \eta) = \frac{1}{2} \sum_p |h^{(p)}(k, \eta)|^2, \quad (18)$$

therefore,

$$|h^{(p)}(k, \eta)| = \frac{C}{\pi} k |h_k(\eta)|. \quad (19)$$

Taking the contribution from each polarization to be same, we get the amplitude of the GWs spectrum as [18]

$$h(k, \eta) = \frac{4l_{pl}}{\sqrt{\pi}} k |h_k(\eta)|. \quad (20)$$

Considering the initial condition as the inflationary stage, the wavelength which crossed over the horizon at time η_i is given by

$$\lambda_i = \frac{2\pi a(\eta_i)}{k} = \frac{1}{H(\eta_i)}, \quad (21)$$

and eq. (1) gives

$$\frac{1}{H(\eta_i)} = \frac{l_0 |\eta_i|^{2+\beta}}{|1+\beta|}.$$

The mode function h_k damps with the expansion of the universe, hence $h_k \propto \frac{1}{a}$. Suppose the initial condition of the mode function is

$$|h_k(\eta_i)| = \frac{1}{a(\eta_i)},$$

then from eqs. (20) - (21), we get

$$h(k, \eta) = 8\sqrt{\pi} \frac{l_{pl}}{\lambda_i}.$$

Again from eq. (21), it follows that $\frac{a'(\eta_i)}{a(\eta_i)} = \frac{k}{2\pi}$. Thus, the initial amplitude of the power spectrum is

$$h(k, \eta_i) = A \left(\frac{k}{k_H} \right)^{2+\beta},$$

where $A = 8\sqrt{\pi} b \frac{l_{pl}}{l_0}$, $b = |1+\beta|^{-(1+\beta)}$, and k_H is the wave number corresponding to the present time.

By matching a and $\frac{a'(\eta)}{a(\eta)}$ at the joining points, we have [5]

$$l_0 = l_H b \zeta_E^{-(2+\beta)} \zeta_2^{\frac{\beta-1}{2}} \zeta_s^\beta \zeta_1^{\frac{\beta-\beta_s}{1+\beta_s}}, \quad (22)$$

where $\zeta_E \equiv \frac{a(\eta_H)}{a(\eta_E)}$, $\zeta_2 \equiv \frac{a(\eta_E)}{a(\eta_2)}$, $\zeta_s \equiv \frac{a(\eta_2)}{a(\eta_s)}$, $\zeta_1 \equiv \frac{a(\eta_s)}{a(\eta_1)}$.

The amplitude of the GWs for the sequential range of comoving wave numbers corresponding to the various evolutionary stages of the universe

(eqs. (1) - (5)) are given by [5, 18]:

$$h(k, \eta_H) = A \left(\frac{k}{k_H} \right)^{2+\beta}, k \leq k_E, \quad (23)$$

$$h(k, \eta_H) = A \left(\frac{k}{k_H} \right)^{\beta-1} \frac{1}{(1+z_E)^3}, k_E \leq k \leq k_H, \quad (24)$$

$$h(k, \eta_H) = A \left(\frac{k}{k_H} \right)^{\beta} \frac{1}{(1+z_E)^3}, k_H \leq k \leq k_2,$$

$$h(k, \eta_H) = A \left(\frac{k}{k_H} \right)^{1+\beta} \left(\frac{k_H}{k_2} \right) \frac{1}{(1+z_E)^3}, \quad (25)$$

$$h(k, \eta_H) = A \left(\frac{k}{k_H} \right)^{1+\beta-\beta_s} \left(\frac{k_s}{k_H} \right)^{\beta_s} \left(\frac{k_H}{k_2} \right) \frac{1}{(1+z_E)^3}, k_2 \leq k \leq k_s, \quad (26)$$

The normalization factor A is determined from the observed CMB anisotropies at lower multipoles, i.e., $\frac{\Delta T}{T} \simeq 0.37 \times 10^{-5}$ at $l \sim 2$ [18]. For the calculation of the factor $(1+z_E)$, (the subscript indicates matter-radiation equality) the redshift corresponding to η_E , we assume $\Omega_\Lambda = 0.7$ and $\Omega_m = 0.3$, thus we get,

$$\frac{a(\eta_H)}{a(\eta_E)} = (1+z_E) = \left(\frac{\Omega_\Lambda}{\Omega_m} \right)^{\frac{1}{3}} \simeq 1.326.$$

The wave number k is proportional to the frequency ν , hence the ratios of the wave numbers can be replaced by the ratios of the frequencies. The Hubble frequency is $\nu_H = \frac{1}{t_H} \simeq 2 \times 10^{-18}$ Hz. The other values of frequency ν are $\nu_E = 1.5 \times 10^{-18}$ Hz, $\nu_2 = 117 \times 10^{-18}$ Hz and $\nu_s = 10^8$ Hz for definiteness, and $\nu_1 = 3 \times 10^{10}$ Hz corresponds to the highest frequency at which the spectral energy density $\Omega_{GW}(\nu)$ does not exceed the nucleosynthesis bound ($\sim 10^{-6}$) [18].

The spectral energy density $\Omega_{GW}(\nu)$ for gravitational wave is defined as:

$$\frac{\rho_g}{\rho_c} = \int \Omega_{GW}(\nu) \frac{d\nu}{\nu},$$

where ρ_g is the energy density of the gravitational waves and ρ_c is the critical energy density of the universe and,

$$\Omega_{GW}(\nu) = \frac{\pi^2}{3} h^2(\nu) \left(\frac{\nu}{\nu_H} \right)^2. \quad (27)$$

4 GW spectrum in squeezed vacuum state

The primordial gravitational waves are created due to the zero-point quantum oscillations which occurred in the early universe [4, 19]. The initial vacuum state with no graviton evolves into multi-particle quantum state through parametric amplification. Hence the primordial gravitational waves are to be considered in the squeezed vacuum state [9, 20].

The squeezed vacuum state is defined as [6, 21]

$$|\xi\rangle = S(\xi)|0\rangle, \quad (28)$$

where $S(\xi)$ is the single mode squeezing operator and is given by

$$S(\xi) = \exp \left[\frac{1}{2} \xi^* c^2 - \frac{1}{2} \xi c^{\dagger 2} \right], \quad (29)$$

where $\xi = r_s e^{i\gamma}$ is a complex number, r_s is the squeezing parameter and γ is the squeezing angle. The unitary transformations of the squeezing operator S on the annihilation and creation operators lead to:

$$S^\dagger(\xi) c S(\xi) = c \cosh r_s - c^\dagger e^{i\gamma} \sinh r_s, \quad (30)$$

$$S^\dagger(\xi) c^\dagger S(\xi) = c^\dagger \cosh r_s - c e^{-i\gamma} \sinh r_s. \quad (31)$$

The gravitational wave field can be written in terms of the mode function and the annihilation and creation operators. Using eqs. (7) and (12), taking the contribution from each polarization to be the same,

$$h(\mathbf{x}, \eta) = \frac{C}{a(\eta)(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} d^3\mathbf{k} [\mu_k(\eta) c_k + \mu_k^*(\eta) c_k^\dagger] e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (32)$$

Thus the amplitude of the gravitational waves spectrum gets modified as

$$\begin{aligned} h(k, \eta) &= \frac{C}{2\pi} k |h(k, \eta)| [1 + 2 \sinh^2 r_s \\ &\quad + \sinh 2r_s \cos \left(\gamma + (2 - n_T) \frac{\pi}{2} \right)]^{1/2}, \end{aligned} \quad (33)$$

where the index $n_T = 2\beta + 5$.

Thus the spectral energy density of gravitational waves in the squeezed vacuum state is obtained in terms of frequency as

$$\begin{aligned} \Omega_{GW}(\nu) &= \frac{\pi^2}{3} h^2(\nu) \left(\frac{\nu}{\nu_H} \right)^2 [1 + 2 \sinh^2 r_s \\ &\quad + \sinh 2r_s \cos \left(\gamma + (2 - n_T) \frac{\pi}{2} \right)], \end{aligned} \quad (34)$$

where $h(\nu)$ in eq. (34) represents the spectral amplitude for the different frequency ranges in eqs. (23) - (26). The spectral energy density is estimated

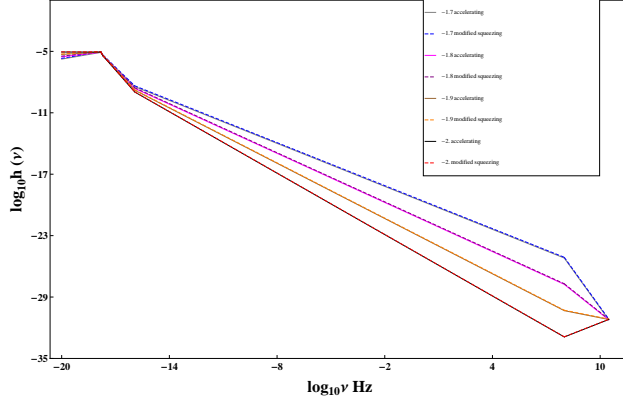


Figure 1: Amplitude of gravitational waves for $\beta = -1.7, -1.8, -1.9$ and -2.0 with zero squeezing (solid lines) and squeezing parameter $r_s = 0.2$ (dashed lines) for the expanding flat FLRW universe.

for various values of the squeezing parameter as well as in the absence of squeezing effect for the accelerating flat FLRW universe and obtained results are:

$$\begin{aligned}
 \Omega_{GW}(\nu) &= 2.34 \times 10^{-6}, \\
 \Omega_{GW}(\nu)(r_s = 0.1) &= 2.54 \times 10^{-6}, \\
 \Omega_{GW}(\nu)(r_s = 0.4) &= 3.78 \times 10^{-6}, \\
 \Omega_{GW}(\nu)(r_s = 0.95) &= 1.037 \times 10^{-5}.
 \end{aligned}
 \tag{35}$$

From the estimate it can be observed that the energy density of gravitational waves does not exceed the upper bound of nucleosynthesis [18].

The gravitational wave spectrum for accelerated flat FLRW universe is studied for various values of β with the squeezing parameter (fig.1, fig. 2 and fig. 3). The obtained spectral amplitudes for both accelerating and decelerating cases are compared with the sensitivity curves of VIRGO, GEO600, LIGO S6 and LISA (fig.2), and Advanced LIGO and Einstein Telescope (fig.3). The spectral energy density of the gravitational waves is found modified due to the squeezed vacuum state (fig. 4).

5 Conclusion

The gravitational waves are very important in understanding the universe. It is believed that the origin of the gravitational waves range from the cosmic inflation period to several astrophysical systems. The gravitational waves have not been detected directly yet but the tremendous in progress in theoretical and observational advancement may help to detect them in nearby

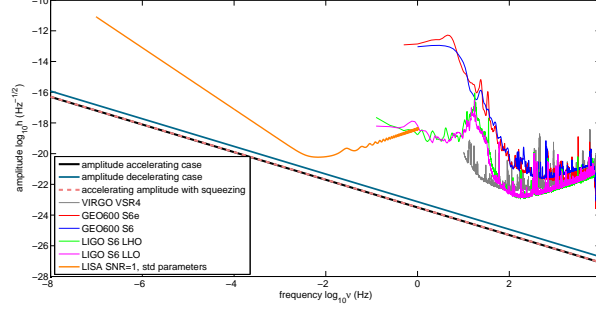


Figure 2: Amplitude of the GWs spectrum with $\beta = -1.9$ for the accelerated (zero squeezing, solid black line and with squeezing parameter $r_s = 0.2$, dashed coral line) and decelerated universe (solid teal line) and sensitivity curves of VIRGO, GEO-600, LIGO and LISA

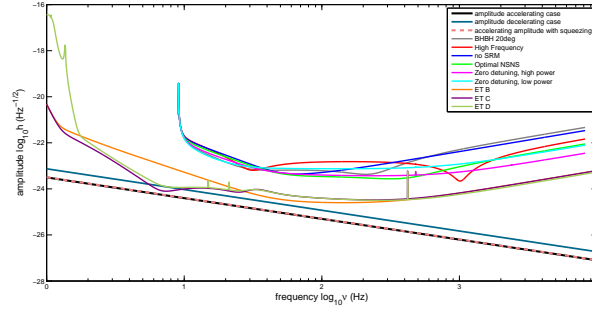


Figure 3: Amplitude of the GWs spectrum with $\beta = -1.9$ for the accelerated (solid black line) and decelerated (solid teal line) universe with the sensitivity of Adv. LIGO and Einstein Telescope, and also with squeezing parameter $r_s = 0.2$ (dashed coral line)

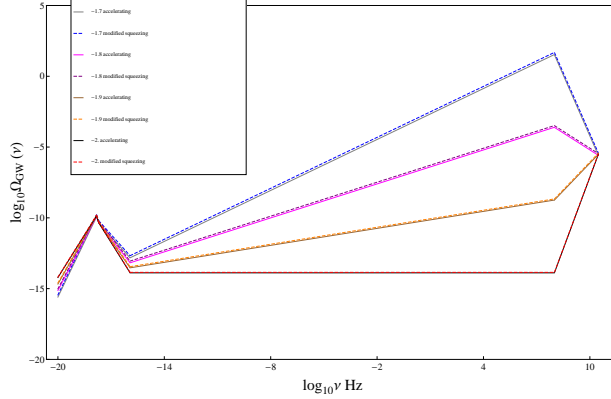


Figure 4: Spectral energy density of gravitational waves for $\beta = -1.7, -1.8, -1.9$ and -2.0 with zero squeezing (solid lines) and with squeezing parameter $r_s = 0.2$ (dashed lines) for the accelerated universe

future. The gravitational waves generated during the inflation are called the primordial gravitational waves and they are believed to be placed in a special quantum state called the squeezed vacuum state which is a well known state in quantum optics. If this is the case, then the quantum effect is expected to be reflected on the amplitude of gravitational waves spectrum. In the present work, the amplitude and energy density of the gravitational waves spectrum are obtained in the squeezed vacuum state for the expanding flat FLRW universe. The obtained gravitational waves spectrum is normalized with WMAP data of CMB. The study shows that there is an enhancement of the amplitude of the spectrum due to the squeezed vacuum effect compared with its zero quantum counter part. Physically the enhancement is arising because of the excitation energy gained by the gravitons due the parametric amplification process on the background metric of early universe. Thus the possibility of detection of the enhanced amplitude of gravitational waves spectrum with the sensitivity of various gravitational wave detectors is examined. It is found that the amplitude of the gravitational waves spectrum gets modified due to the squeezing effect for the accelerated flat FLRW universe and is likely to be detected with the Einstein telescope. Further, it is found that the estimated spectral energy density of the gravitational waves in the squeezed vacuum state does not exceed the nucleosynthesis upper bound.

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References

- [1] B. S. Sathyaprakash, B. F. Schutz, Living Rev. Relativity **12** 2 (2009)
- [2] R. H. Brandenberger, arXiv:hep-ph/9910410v1 (1999)
- [3] A. D. Linde, *Particle Physics and Inflationary Cosmology* (CRC Press, 1990)
- [4] L. P. Grishchuk, Sov. Phys. JETP **40** 409 (1975)
- [5] L. P. Grishchuk, Lect. Notes Phys. **562** 167 (2001)
- [6] B. L. Schumaker, Phys. Rep. **135** 317 (1986)
- [7] J. R. Klauder and B. S. Skagerstam, *Coherent States* (World Scientific, 1985)
- [8] C. M. Caves and D. F. Walls, Phys. Rev. Lett **57** 2164 (1980)
- [9] L. P. Grishchuk, Phys. Rev. D **53** 6784 (1996)
- [10] L. P. Grishchuk, arXiv:0707.3319v4 [gr-qc] (2010)
- [11] L. P. Grishchuk and Yu. V. Sidorov, Phys. Rev. D **42** 3413 (1990)
- [12] J. Abadie et al., arXiv:1203.2674v2 [gr-qc] (2012)
- [13] B. P. Abbott et al., Phys. Rev. D **80** 042003 (2009)
- [14] H. Grote, Class.Quant.Grav. **27** 084003 (2010)
- [15] M. Hewitson et al., Class.Quant.Grav. **20** S581 (2003)
- [16] S. J. Waldman, arXiv:1103.2728v1 [gr-qc] (2011)
- [17] S. Kuroyanagi, T. Chiba and N. Sugiyama, Phys. Rev. D **79** 103501 (2009)
- [18] Y. Zhang, Y. Yuan, W. Zhao and Y. T. Chen, Class. Quant. Grav. **22** 1383 (2005)
- [19] L. P. Grishchuk, arXiv:gr-qc/9302036v1 (1993)
- [20] S. Nakamura, N. Yoshino and S. Kobayashi, Prog. Theor. Phys. **88** 1107 (1992)
- [21] D. F. Walls, Nature **300** 141 (1983)